the system is stable provided $(C-A+c_2-b_2)$ and $(C-B+c_1-b_1)$ are negative. Furthermore, K will be positive definite if

$$C-A+c_2-b_2>0, \qquad C-B+c_1-b_1>0$$

$$N<\left(\frac{-(k_{11}+k_{22})}{(C-B+c_1-b_1)}\right)^{1/2} \tag{16}$$

or, if

$$C-A+c_2-b_2<0, \qquad C-B+c_1-b_1<0$$

$$N<\left(\frac{-(k_{12}+k_{21})}{(C-A+c_2-b_2)}\right)^{1/2} \tag{17}$$

Either set of inequalities yields a stable operation and both follow directly from the positive definiteness of the matrix K in Eq. (1). They also represent the second and third situations discussed in Ref. 1, where it is required to calculate the roots of the stability boundary equation to arrive at the same conclusions. It should be noted that these situations occur when the rotor does not spin about a principal axis of minimum or maximum inertia.

In addition to the three cases analyzed in Ref. 1, there is yet another range of permissible rotor speeds and parameter values which insures stable DTG operation. This situation is identified by exercising the new matrix condition given in Sec. III, namely, $\det(\tilde{K})>0$ and trace $(4\tilde{K}-\tilde{G}^2)>0$, with $\tilde{K}$ negative definite. This requires that Eq. (1) be transformed into the system described by Eq. (2). Requiring $\det(\tilde{K})>0$ and $\tilde{K}$ to be negative definite yields

$$k_{11}+k_{22}+2N^2(C-B+c_1-b_1)<0 \tag{18}$$

$$k_{12}+k_{21}+2N^2(C-A+c_2-b_2)<0 \tag{19}$$

while the trace of $4\tilde{K}-\tilde{G}^2$ is given by

$$4\frac{k_{11}+k_{22}+N^2(C-B+c_1-b_1)}{A+a_1}+2\frac{N^2(A+B-C)^2}{(A+a_1)(B+a_2)}$$
$$+4\frac{k_{12}+k_{21}+N^2(C-A+c_2-b_2)}{B+a_2} \tag{20}$$

Demanding that Eq. (20) be positive yields

$$4(k_{11}+k_{22})(B+a_2)+4(k_{12}+k_{21})(A+a_1)$$
$$+2N^2[(A+B-C)^2$$
$$+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)]>0 \tag{21}$$

which results in two subcases. The first case is

$$(A+B-C)^2+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)>0 \tag{22}$$

and the DTG is stable for all values of the rotor speed N. The other subcase occurs when the inequality sign in Eq. (22) is reversed. For this condition, the DTG is stable, provided that N is such that

$$N<[-2[(k_{11}+k_{22})(B+a_2)+(k_{12}+k_{21})(A+a_1)]^{1/2}$$
$$\div[(A+B-C)^2+2(A+a_1)(C-A+c_2-b_2)$$
$$+2(B+a_2)(C-B+c_1-b_1)] \tag{23}$$

It should be noted that the existence of additional stable regions for the case when $\tilde{K}$[from Eq. (2)] is not negative definite is a result of the stabilizing effect of gyroscopic forces.

VI. Conclusion

A simple matrix stability condition is derived for linear gyroscopic systems with two degrees of freedom. The matrix method of analysis requires less calculation than the usual procedures of generating the system response or solving the associated eigenvalue problem. Also, matrix techniques have the ability to indicate subtleties in the design choices which otherwise may be undiscovered. The new matrix condition along with standard matrix stability requirements are applied to a model of a dynamically tuned gyroscope. It is shown that the matrix approach yields a larger range of parameter values to choose from for the stable dynamically tuned gyroscope operation than previously indicated in the literature. It is hoped that the new matrix condition may find application in the general class of complex gyroscopic systems which includes the DTG.

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Model Reduction of Control Systems

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Introduction

THE analysis and design of large-order control systems is quite tedious and costly. Therefore, it is desirable to replace a given large-order system with a lower-order system in such a way that the lower-order system retains the significant characteristics of the given system.

From the standpoint of flying qualities, the usefulness of the model reduction technique lies in verifying compliance with the specifications. The analytical description of a typical augmented aircraft system is usually quite complicated. Thus, it becomes difficult to check whether or not the designed

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aircraft system meets the flying qualities specifications. What is needed to do this is a reduced-order model of a suitable order that depends on the aircraft response being checked. The flying qualities requirements are then compared with the parameters of this reduced-order model. During the past two decades, several papers have tried to tackle the model reduction problem. The method based on retaining dominant poles[1] was followed by the method of continued fractions.[2] The frequency domain techniques[3,4] based on optimization are either restricted[3] to a certain class of transfer functions or do not give a good approximation.[4] The technique of the Caucer continued-fraction expansion of a given transfer function about an arbitrary point and also about infinity was proposed by Parthasarthy and Jayashimaha,[5] while the technique of matching Taylor series expansion coefficients was proposed by Lepschy et al.[6] The technique proposed in this Note is in the frequency domain based on optimization and is a modification of the technique of Rao and Lamba.[3]

The technique of Rao and Lamba is quite effective; but as formulated, it cannot be applied to those transfer functions that have either a zero or a pole at the origin. The transfer function related to the AFTI/F-16 aircraft (Advanced Fighter Technology Integration/F-16) that the author reduced had a zero at the origin. This necessitated the modification proposed here. The modified technique is quite general and is applicable to any transfer function, whether it has a zero or a pole at the origin.

Under the proposed technique, the parameters of the lower-order transfer function are determined by minimizing a weighted mean-square error between the frequency responses of the given and reduced systems.

Formulation of the Problem

Let the transfer fucntion of a given single variable continuous control system be given by

$$F(s)=\sum_{i=0}^{p} a_i s^i \Big/ \sum_{i=0}^{q} b_i s^i, \quad p\leq q \tag{1}$$

where a_i and b_i are known real constants. The transfer function $F(s)$ is to be approximated by a lower-order transfer function $G(s)$ given by

$$G(s)=\sum_{i=0}^{m} c_i s^i \Big/ 1+\sum_{i=1}^{n} d_i s^i, \quad m\leq n<q \tag{2}$$

where c_i and d_i are unknown real parameters to be determined. It is required to determine c_i and d_i so that the frequency responses of $F(s)$ and $G(s)$ are as "close" as possible.

Derivation of the Design Algorithm

The frequency responses of the given and the reduced control systems are given by

$$F(j\omega)=\frac{L+j\omega M}{R+j\omega S}, \qquad G(j\omega)=\frac{P+j\omega Q}{\sigma+j\omega\tau} \tag{3}$$

$$L=a_0-a_2\omega^2+a_4\omega^4-\ldots$$
$$M=a_1-a_3\omega^2+a_5\omega^4-\ldots$$
$$R=b_0-b_2\omega^2+b_4\omega^4-\ldots$$
$$S=b_i-b_3\omega^2+b_5\omega^4-\ldots$$
$$P=c_0-c_2\omega^2+c_4\omega^4-\ldots$$
$$Q=c_1-c_3\omega^2+c_5\omega^4-\ldots$$
$$\sigma=1-d_2\omega^2+d_4\omega^4-\ldots$$
$$\tau=d_1-d_3\omega^2+d_5\omega^4-\ldots \tag{4}$$

Define the error function between the frequency responses of $F(s)$ and $G(s)$ as

$$E=\int_{\omega_0}^{\omega'}\left|\frac{L+j\omega M}{R+J\omega S}-\frac{P+j\omega Q}{\sigma+j\omega\tau}\right|^2 \mathrm{d}\omega \tag{5}$$

where $[\omega_0,\omega']$ is the frequency interval of interest over which E will be minimized. Equating to zero the partial derivatives of E with respect to the unknown parameters c_i and d_i results in nonlinear algebraic equations involving the unknown parameters. These nonlinear equations can be quite difficult to solve and can take a lot of computer time. Following Rao and Lamba,[3] a suitable modification in the error function given by Eq. (5) is made in such a way that the accuracy of approximation is retained and linear equations involving the unknown parameters are obtained. The error function is modified by taking the product of the denominators of the given and reduced transfer functions as a weighting function. Experience in working with this "linearizing technique" has shown that this simplifies the problem tremendously and gives fairly accurate results. Thus, the modified error function is defined as

$$E_m=\int_{\omega_0}^{\omega'}[(L\sigma-\omega^2M\tau-PR+\omega^2QS)^2 + (\omega L\tau+\omega M\sigma-\omega SP-\omega RQ)^2]\mathrm{d}\omega \tag{6}$$

Equating to zero the partial derivatives of E_m given by Eq. (6) with respect to $c_0,c_1,c_2,\ldots$ and $d_1,d_2,d_3,\ldots,$ yields after considerable manipulation

$$A\cdot U=B \tag{7}$$

where

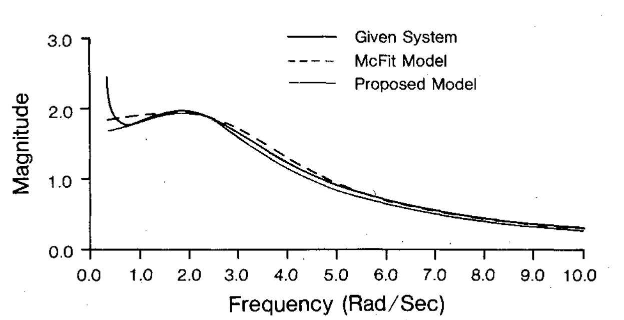

Fig. 1 Comparison of the magnitude of the proposed model with that of McFIT and the given system.

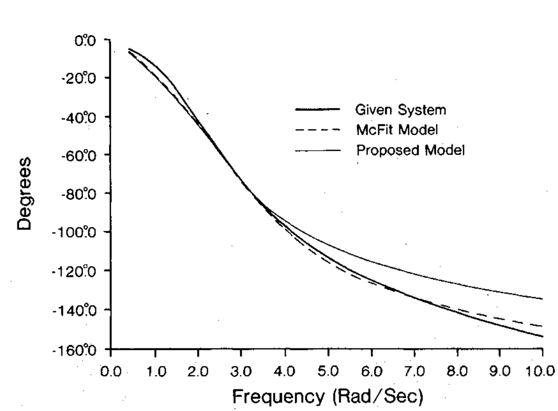

Fig. 2 Comparison of phase of the proposed model with that of McFIT and the given system.

$$A=\begin{bmatrix}
T_0+V_2 & 0 & -(T_2+V_4) & 0 & \cdots & U_2-\lambda_2 & A_2+B_4 & -(U_4-\lambda_4) & -(A_4+B_6) & \cdots \\
0 & T_2+V_4 & 0 & -(T_4+V_6) & \cdots & -(A_2+B_4) & U_4-\lambda_4 & A_4+B_6 & -(U_6-\lambda_6) & \cdots \\
-(T_2+V_4) & 0 & T_4+V_6 & 0 & \cdots & -(U_4-\lambda_4) & -(A_4+B_6) & U_6-\lambda_6 & A_6+B_8 & \cdots \\
0 & -(T_4+V_6) & 0 & T_6+V_8 & \cdots & A_4+B_6 & -(U_6-\lambda_6) & -(A_6+B_8) & U_8-\lambda_8 & \cdots \\
\cdots & & & & & & & & & \cdots \\
\cdots & & & & & & & & & \cdots \\
\cdots & & & & & & & & & \cdots \\
U_2-\lambda_2 & -(A_2+B_4) & -(U_4-\lambda_4) & A_4+B_6 & \cdots & D_4+E_2 & 0 & -(D_6+E_4) & 0 & \cdots \\
A_2+B_4 & U_4-\lambda_4 & -(A_4+B_6) & -(U_6-\lambda_6) & \cdots & 0 & D_6+E_4 & 0 & -(D_8+E_6) & \cdots \\
-(U_4-\lambda_4) & A_4+B_6 & U_6-\lambda_6 & -(A_6+B_8) & \cdots & -(D_6+E_4) & 0 & D_8+E_6 & 0 & \cdots \\
\cdots & & & & & & & & & \\
\cdots & & & & & & & & & \\
\cdots & & & & & & & & &
\end{bmatrix} \tag{8}$$

$$U=(c_0,c_1,c_2,c_3,\ldots;\ d_1,d_2,d_3\ldots)^T$$

$$B=(A_0+B_2,U_2-\lambda_2,-(A_2+B_4),-(U_4-\lambda_4),\ldots;\ 0,D_4+E_2,0,\ldots)^T \tag{9}$$

$$A_r=\int_{\omega_0}^{\omega'} LR\omega^r \mathrm{d}\omega,\quad B_r=\int_{\omega_0}^{\omega'} S\cdot M\omega^r \mathrm{d}\omega$$

$$U_r=\int_{\omega_0}^{\omega'} R\cdot M\omega^r \mathrm{d}\omega,\ V_r=\int_{\omega_0}^{\omega'} S^2\cdot\omega^r \mathrm{d}\omega$$

$$\lambda_r=\int_{\omega_0}^{\omega'} S\cdot L\omega^r \mathrm{d}\omega,\quad T_r=\int_{\omega_0}^{\omega'} R^2\cdot\omega^r \mathrm{d}\omega$$

$$E_r=\int_{\omega_0}^{\omega'} L^2\omega^r \mathrm{d}\omega,\quad D_r=\int_{\omega_0}^{\omega'} M^2\omega^r \mathrm{d}\omega \tag{10}$$

Equation (7) can be solved for the unknown vector U as

$$U=A^{-1}\cdot B \tag{11}$$

Equation (11) gives the values of the unknown parameters c_i and d_i that, when substituted into Eq. (2), completely determine the reduced-order transfer function.

A Numerical Example and Comparison of Results

In this section a numerical example related to the aircraft pitch rate/pilot input of the AFTI/F-16 aircraft is simplified using the proposed techniqe. The transfer function considered is a simplification of the complete AFTI/F-16 flight control block diagram in the original q reconfiguration mode. The transfer function relating the aircraft pitch rate and the pilot input is given by

$$\begin{aligned}F(s) = [&(0.1402E+08)s^8+(0.9421E+09)s^7\\
&+(0.2377E+11)s^6+(0.2743E+12)s^5+(0.1381E+13)s^4\\
&+(0.2233E+13)s^3+(0.8755E+12)s^2+(0.3988E+11)s]/\\
[&s^{13}+246.7s^{12}+(0.2747E+05)s^{11}+(0.1703E+07)s^{10}\\
&+(0.6162E+08)s^9+(0.1323E+10)s^8+(0.1685E+11)s^7\\
&+(0.1252E+12)s^6+(0.5264E+12)s^5+(0.1222E+13)s^4\\
&+(0.1524E+13)s^3+(0.5866E+12)s^2+(0.6227E+11)s\\
&+(0.1731E+11)]\end{aligned} \tag{12}$$

Using Eq. (11) with $\omega_0=0.3$ and $\omega'=1.5$, the reduced-order model with one zero and three poles is given by

$$G_1(s)=\frac{0.6418s+1.654}{0.01764s^3+0.2103s^2+0.601s+1} \tag{13}$$

Using the McFIT program based on the search technique of Rosenbrock currently being used in industry, the reduced-order model with one zero, two poles, and a delay element is given by

$$G_2(s)=\frac{e^{-0.0736s}(2.31598608s+20.6998258)}{s^2+4.168408575s+11.147204} \tag{14}$$

Since a delay factor can be approximated by a pole, the reduced model obtained by the proposed technique and given by Eq. (13) would be comparable to the one obtained by the McFIT technique and given by Eq. (14). Figures 1 and 2 give the frequency responses. From the point of view of the flying qualities, the frequency range of interest for this example is approximately 0.3-10 rad/s. It is clear from the figures that the proposed technique gives a fairly good fit compared with the one given by the McFIT method. The main advantage of the proposed technique over the McFIT program is that it provides significant savings in computer time. It also gives unique and, therefore, repeatable results.

Summary

In this Note a computer-aided method of simplifying single-variable control systems is developed. It is a modification of the technique of Rao and Lamba that can be applied directly to control systems with either a pole or a zero at the origin.

The McFIT model reduction technique being used in industry is iterative in nature and can take a large number of iterations before a solution begins to converge. The proposed method is a one-step procedure and thus provides significant savings in comptuer time. It also gives unique, and therefore repeatable, results.

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Optimal Many-Revolution Orbit Transfer

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Introduction

THE problem of low-thrust, minimum-time orbit transfer has been studied analytically by several authors in the last decades. The method of two time variables has been used by Levin,[1] Shi and Eckstein,[2] and Eckstein and Shi[3] to study the effect of ad hoc thrust programs on the orbit without considering optimization. In an important paper, Edelbaum[4] solved the optimal one-orbit control problem for the first time, and obtained a suboptimal solution for the long time scale problem.

In this Note we shall also use the method of two time scales, separating the optimal control problem into a "fast" time scale problem over one orbit, and a "slow" time scale problem over the entire transfer. We shall recapitulate Edelbaum's solution of the fast time scale problem, and then we shall solve, for the first time, the optimal control problem for the overall transfer. The optimal control law for the slow time scale problem will be found in explicit form. This reduces the slow time scale problem to a two-point boundary value problem in semimajor axis inclination space. The solution space for this problem has been globally mapped, and explicit total velocity change requirements for any desired transfer can be easily obtained. The solution assumes constant thrust with decreasing mass of the vehicle, but is also optimum for any slowly varying throttle program.

The Short Time Scale Problem

The optimal control problem over the fast time scale of one orbit was first solved by Edelbaum. However, we quickly review those portions pertinent to our later discussion. For a low-thrust vehicle in a nearly circular orbit, the two Lagrange planetary equations we shall need (in their acceleration component form) are (Danby[5]):

$$\frac{\mathrm{d}a}{\mathrm{d}t}=\frac{2Aa^{3/2}\cos\vartheta}{\sqrt{\mu}} \tag{1}$$

$$\frac{\mathrm{d}i}{\mathrm{d}t}=\frac{Aa^{1/2}\sin\vartheta\cos f}{\sqrt{\mu}} \tag{2}$$

where μ is the gravitational parameter, a the semimajor axis, i the inclination, and f the true anomaly, measured from the node since the eccentricity is small. The vehicle acceleration is A, and the vehicle pitch angle from the orbital plane is ϑ. For small eccentricity there is no dependence on any vehicle yaw component of acceleration.

Assuming the control $\vartheta(f)$ to be a function of true anomaly, we may pose the problem of maximizing the inclination change over one orbit while still achieving a given semimajor axis change Δa. Assuming small changes over the orbit leads to the optimization problem

$$\delta\int_0^{2\pi}\left[\frac{a^2A}{\mu}\sin\vartheta(f)\cos f+\lambda\left(\frac{2a^3A}{\mu}\cos\vartheta(f)-\frac{\Delta a}{2\pi}\right)\right]\mathrm{d}f=0 \tag{3}$$

Simple techniques yield the control law

$$\vartheta(f)=\tan^{-1}\left(\frac{\cos f}{\sqrt{1/u-1}}\right) \tag{4}$$

and the changes in a and i per orbit are

$$\mathrm{d}a=\frac{8a^3A}{\mu}\sqrt{1-u}\,K(u) \tag{5}$$

$$\mathrm{d}i=\frac{4a^2A}{\mu}\left[\frac{1}{\sqrt{u}}E(u)+\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)K(u)\right] \tag{6}$$

Here, in anticipation of the next section, we have introduced the control variable

$$u\equiv\frac{1}{4\lambda^2a^2+1} \tag{7}$$

The operative range of u is from 0 to 1, and K and E are the complete elliptic integrals of the first and second kinds, respectively. Very convenient approximation formulas for these functions are given by Abramowitz and Segun.[6]

Several other results must be mentioned. The net change in eccentricity and node per orbit is zero with this control program, so initially circular obits stay nearly circular. The control law $\vartheta(f)$ varies from a pure in-track acceleration for $u=0$ when only a changes and to a square wave for $u=1$ when only the inclination changes.

The Long Time Scale Problem

In the previous section we reviewed the optimal way to produce small changes in the orbital elements over one orbit. On the long time scale of the entire transfer, these expressions can be divided by the Keplerian period $\mathrm{d}t=2\pi a^{3/2}/\sqrt{\mu}$ (which is to be regarded as a "short" time) to yield equations of motion on the long time scale. Also on the long time scale we must include the effects of depletion of fuel which (for constant thrust) causes the vehicle acceleration to vary as

$$A(t)=\frac{A_0}{1-\dot{m}t} \tag{8}$$

Here $\dot{m}$ is the specific fuel usage rate, and A_0 is the initial acceleration. This last effect introduces explicit time

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